The Szemerédi–Trotter Theorem

Incidences between Points and Lines

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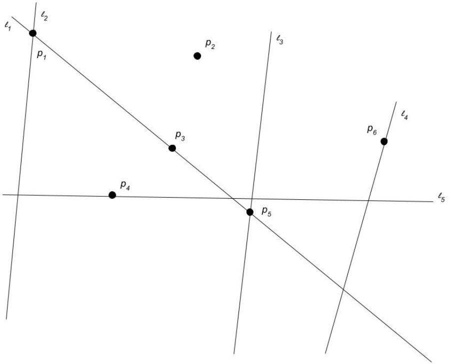
Setup: Points and Lines

Suppose we are given n points, P, and m lines, $\mathcal{L}$.



Here, $n = 6$, $m = 5$, and notice that there are 7 pairs of points and lines, (p, ℓ), such that the point p lies on ℓ. Such pairs are known as *incidences*.

Setup: Incidences

Question

Suppose we are given n points, P, and m lines, $\mathcal{L}$, in Euclidean space. How large can the following be:

$$\mathcal{I}(P, \mathcal{L}) := \#\{(p, \ell) \in P \times \mathcal{L} : p \in \ell\}?$$

Remark: We suppose without loss of generality that every line contains at least one point.

Incidences: Known Bounds

Question

Suppose we are given n points, P, and m lines, $\mathcal{L}$, in Euclidean space. How large can the following be:

$$\mathcal{I}(P, \mathcal{L}) := \#\{(p, \ell) \in P \times \mathcal{L} : p \in \ell\}?$$

There are a few key bounds we can know.

- Trivial bound: $\mathcal{I}(P, \mathcal{L}) \leq nm$.
- Cauchy–Schwarz: $\mathcal{I}(P, \mathcal{L}) \leq \sqrt{n}(nm + m^2)^{1/2}$
- Szemerédi–Trotter: $\mathcal{I}(P, \mathcal{L}) \leq C(n^{2/3}m^{2/3} + n + m)$

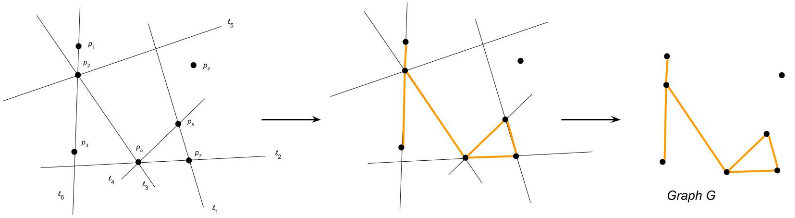
We will spend this presentation outlining one way to prove the Szemerédi–Trotter bound.

Outline

From Points and Lines to a Graph

We transform our points P and our lines $\mathcal{L}$ into a graph $G = (V, E)$.

- We let our points P become our vertices V (i.e. $|V| = n$), and
- An edge between p_i and p_j is formed if p_i and p_j are on a common line $\ell \in \mathcal{L}$ and the points p_i and p_j are adjacent on ℓ.

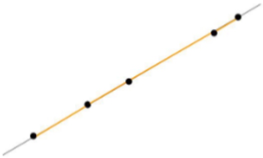


Counting Edges

Lemma

Let $G = (V, E)$ be the graph obtained from the previous slide on n points and m lines. Then, $|E| = \mathcal{I}(P, \mathcal{L}) - m$.

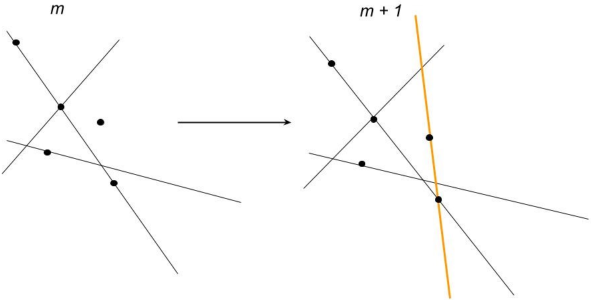
Proof: We follow by induction on m. If there is one line, ℓ_1, then notice that the number of edges is always one less than the number of incidences.



In the above, we have $|E| = 4$, $\mathcal{I}(P, \ell_1) = 5$, and $m = 1$.

$|E| = \mathcal{I}(P, \mathcal{L}) - m$ **ctd.**

We now assume the proposition holds for m lines, and show it holds for $m + 1$ lines.



From the first m lines we obtain $\mathcal{I}(P, \mathcal{L} \setminus \ell_{m+1}) - m$ many edges, and from the new (orange) line we obtain $\mathcal{I}(P, \ell_{m+1}) - 1$ many edges from the base case. Hence, $|E| = \mathcal{I}(P, \mathcal{L}) - (m + 1)$. □

Developing Graph Theory

At this point, we have turned our set of n points and m lines into a graph $G = (V, E)$. In particular, we know

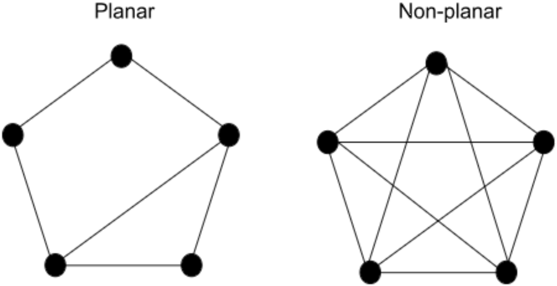
$$|V| = n \quad \text{and} \quad |E| = \mathcal{I}(P, \mathcal{L}) - m.$$

We will develop some graph theory to give us inequalities relating $|V|$ and $|E|$ to one another.

Graph Theory: Planar

Definition (Planar)

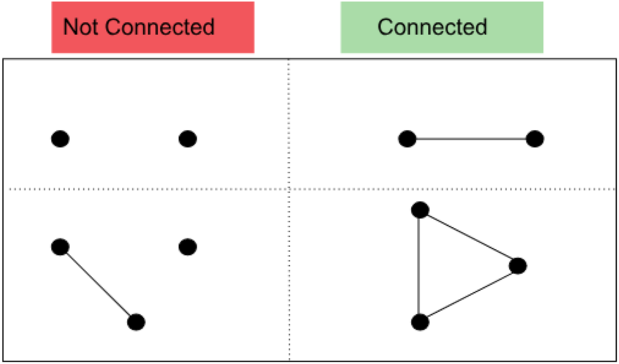
A graph G is planar if we can draw G such that edges only intersect at their endpoints (i.e. no edges cross each other).



Graph Theory: Connected

Definition (Connected)

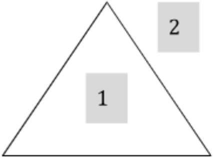
A graph G is connected if, for each pair of vertices, there at least one path which joins them.



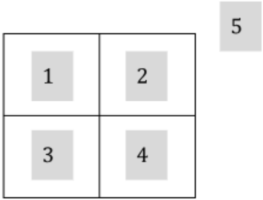
Graph Theory: Faces

Definition (Face)

A face in a planar graph G is a region bounded by a set of edges and vertices in the embedding.



2 faces



5 faces

Euler's Theorem

Theorem (Euler's Theorem)

Let G be a planar connected graph. Then,

$$V(G) - E(G) + F(G) = 2.$$

Proof Outline: We do induction on $E(G)$.

- **Base Case**: $E(G) = 1$: There are only two possibilities.
- **Inductive Step**: Assume true for $E(G) = k$ and prove true for $E(G) = k + 1$.
  - If possible: remove one edge and use inductive hypothesis.
  - Otherwise, G is a "tree" and we can prove this directly for trees.

□

Euler's Theorem ctd.

More generally,

Theorem

Let G be a planar graph. Then,

$$V(G) - E(G) + F(G) = k(G) + 1$$

where $k(G)$ is the number of connected components of G.

Graph Inequalities

Lemma

If G is planar, then $|E| - 3|V| \leq 0$.

Proof: From Euler's theorem:

$$V - E + F \geq 0.$$

Furthermore, $F \leq \frac{2}{3}E$. Combining these and rearranging the inequality gives the desired result. □

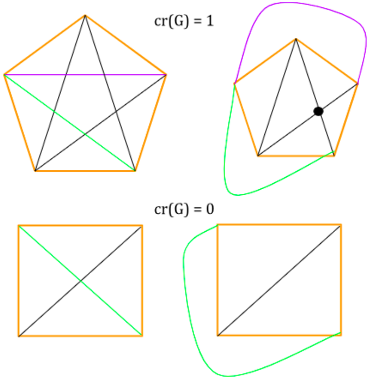
As we cannot guarantee a planar graph, we must obtain a more general result relating edges and vertices, which can be done once we define *crossing numbers*.

Crossing Number $cr(G)$

Definition (Crossing Number)

Given a graph G, we define $cr(G)$ to be the minimum number of edge crossings achievable when laying out G in the 2D plane.

For example: G is planar if and only if $cr(G) = 0$.



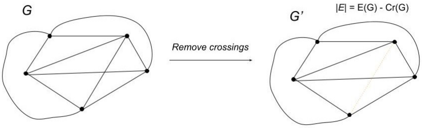
Graph Inequalities ctd.

Lemma

If G is a graph, then

$$|E| - 3|V| \leq cr(G).$$

Proof: If $cr(G) = 0$ then G is planar and we already proved the result. Otherwise, we remove edges to make G planar.



Hence,

$$|E'| - 3|V'| \leq 0 \implies (|E| - cr(G)) - 3|V| \leq 0.$$

□

The Crossing Number Lemma

As it turns out, the previous inequality is not as good as we need for the proof of Szemerédi–Trotter. But, one can use *probability* to improve the inequality, and obtain the following result.

Lemma (Crossing Number Lemma)

Let $G = (V, E)$ be a graph with $|E| \geq 4|V|$. Then,

$$cr(G) \geq \frac{|E|^3}{64|V|^2}.$$

Proving Szemerédi–Trotter

Recall that we have taken our n points and m lines and turned it into a graph $G = (V, E)$ such that

$$|V| = n \quad \text{and} \quad |E| = \mathcal{I}(P, \mathcal{L}) - m.$$

Furthermore, recall that we want to prove

$$\text{Szemerédi–Trotter} : \mathcal{I}(P, \mathcal{L}) \leq C(n^{2/3}m^{2/3} + n + m).$$

We break into cases based on the Crossing Number Lemma.

Case 1: $|E| < 4|V|$. Then,

$$\mathcal{I}(P, \mathcal{L}) = |E| + m < 4n + m.$$

Proving Szemerédi–Trotter ctd.

Case 2: $|E| \geq 4|V|$. Then, by the Crossing Number Lemma,

$$\frac{|E|^3}{64|V|^2} \leq cr(G).$$

By construction, $cr(G) \leq m^2$.

Rearranging and plugging in $|E|$ and $|V|$, we see

$$\mathcal{I}(P, \mathcal{L}) \leq 4n^{2/3}m^{2/3} + m.$$

Adding these two cases together, we obtain

$$\mathcal{I}(P, \mathcal{L}) \leq 4(n^{2/3}m^{2/3} + n + m).$$

□

Citations

We used the following texts in our studies:

(1) Miklós Bóna: *A Walk Through Combinatorics*

(2) Larry Guth: *The Polynomial Method: Lecture Notes*

(3) Alex Iosevich: *A View From the Top*

Thank you for listening!

Any Questions?